Homotopy Groups of Concordance

Riley Moriss

February 3, 2026

We are going to tabulate the computed homotopy groups of concordance spaces of the disc as far as we can find them / compute them. We will do this in the stable range. Recall that in the stable range, $i << n$, we have a series of isomorphisms

$$\pi_{i-1}\mathscr{C}(D^n) \cong \pi_i B\mathscr{C}(D^n) \cong \pi_i H(D^n) \cong \pi_i \mathcal{H}(D^n) \cong \pi_i \Omega \mathrm{Wh}^{\mathrm{Diff}}(*) \cong \pi_{i+1}\mathrm{Wh}^{\mathrm{Diff}}(*)$$

which we will use to compute these groups, but moreover which add to the interest of these groups overall. In particular we are tabulating all of these groups stably. To perform the computations we will need a couple of lemmas

Lemma ([Wal82], Remark.Cor 3.4)**.** *For all primes p and natural numbers $j < 2p-3$ there is an isomorphism*

$$\pi_j \mathrm{Wh}^{\mathrm{Diff}}(*)_{(p)} \cong K_j(\mathbb{Z})_{(p)}$$

Since the K theory of the integers is mostly computed for low j this means that we can just compare locally to these groups, for instance listed in [Wei, Table 5.1]. For the primes that are not covered by this theorem we will need the work of [Rog03][Rog02]. Rognes gives us the values for regular primes, the smallest of which is 37. Thus the bound $2p-3 = 2(37)-3 = 71$ means that we can only compute the torsion for all primes for the first 70 homotopy groups of the Whitehead space, or whats the same the first 68 homotopy groups of the concordance space. After this we would get the torsion for primes > 37 and need to know the torsion for primes ≤ 37 by other means. However since 37 is not regular it is not covered by Rognes and so we do not know it by other means. In the end this is mostly irrelevant because to find out the two torsion we also clearly need Rognes and he only provides it up to π_{19} (after that it is extension problems or unknown). Thus to summarise how we will compute these groups we can rewrite them as follows

$$\pi_i \mathrm{Wh}^{\mathrm{Diff}}(*) \cong \bigoplus_{p \leq \frac{i+3}{2}} \pi_i \mathrm{Wh}^{\mathrm{Diff}}(*)_{(p)} \bigoplus_{p > \frac{i+3}{2}} K_i(\mathbb{Z})_{(p)}.$$

We will leave subtracting two from the index in the table to the reader.

1 The Computations

We are borrowing the notation (0?) form Weibel, it is to indicate a group that is conjecturally zero but unknown. I believe that it has been checked to not contain prime factor for like the first 10 million primes however, so if this group is non-zero it is 'large" in the sense that its prime factors are large, it is however finite.

[Rog03, Example 4.8] kindly lists the low dimensional torsion explicitly for primes $3, 5$, we need to unpack [Rog03, Cor 4.9] for the primes $7, 11$. Luckily the bounds are growing quickly, for prime 11

i	$\frac{i+3}{2}$	Largest prime $p \leq \frac{i+3}{2}$	$\pi_i \mathrm{Wh}^{\mathrm{Diff}}(*)_{(2)}$	$\bigoplus_{2<p\leq \frac{i+3}{2}} \pi_i \mathrm{Wh}^{\mathrm{Diff}}(*)_{(p)}$	$\bigoplus_{p>\frac{i+3}{2}} K_i(\mathbb{Z})_{(p)}$
0	1.5	–	0	0	$\mathbb{Z}$
1	2	2	0	0	0
2	2.5	2	0	0	0
3	3	3	$\mathbb{Z}_2$	0	$K_3 = \mathbb{Z}_{48} \implies 0$
4	3.5	3	0	0	0
5	4	3	$\mathbb{Z}$	0	$\mathbb{Z}$
6	4.5	3	0	0	0
7	5	5	$\mathbb{Z}_2$	0	$K_7 = \mathbb{Z}_{240} \implies 0$
8	5.5	5	0	0	(0?)
9	6	5	$\mathbb{Z}_2 \oplus \mathbb{Z}$	0	$\mathbb{Z}$
10	6.5	5	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_8$	0	0
11	7	7	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$K_{11} = \mathbb{Z}_{1008} \implies 0$
12	7.5	7	$\mathbb{Z}_4$	0	(0?)
13	8	7	$\mathbb{Z}$	$\mathbb{Z}_7$	$\mathbb{Z}$
14	8.5	7	$\mathbb{Z}_4$	$\mathbb{Z}_3 \oplus \mathbb{Z}_9$	0
15	9	7	$\mathbb{Z}_2^2$	0	$K_{15} = \mathbb{Z}_{480} \implies 0$
16	9.5	7	$\mathbb{Z}_2 \oplus \mathbb{Z}_8$	$\mathbb{Z}_3$	(0?)
17	10	7	$\mathbb{Z}_2^2 \oplus \mathbb{Z}$	0	$\mathbb{Z}$
18	10.5	7	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_{32}$	$\mathbb{Z}_3 \oplus \mathbb{Z}_5$	0
19	11	11	$\mathbb{Z}_2 \rtimes \mathbb{Z}_2 \rtimes \mathbb{Z}_8 \rtimes \mathbb{Z}_2$	$\mathbb{Z}_7$	$K_{19} = \mathbb{Z}_{528} \implies 0$

the lowest bound that he gives is 20 and hence there is no 11 torsion in the groups tabulated. On the other hand there are two values that lie in his bounds for the prime 7 and are less than or equal to 19 and those are $13, 19$ hence we have a $\mathbb{Z}_7$ in those degrees.

Finally note that the $\mathbb{Z}$ factors that appear accross the columns are all the same $\mathbb{Z}$, that is there is at most one $\mathbb{Z}$ factor in the final homotopy group, we know it is rationally at most a $\mathbb{Q}$. Thus summing across the rows of our table gives us the final groups.

Table 1: Comparison of Homotopy Groups ($i = 0$ to 6)

$\pi_0 \mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_1 \mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_2 \mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_3 \mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_4 \mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_5 \mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_6 \mathrm{Wh}^{\mathrm{Diff}}(*)$
$--$	$\pi_0 \mathcal{H}(D^n)$	$\pi_1 \mathcal{H}(D^n)$	$\pi_2 \mathcal{H}(D^n)$	$\pi_3 \mathcal{H}(D^n)$	$\pi_4 \mathcal{H}(D^n)$	$\pi_5 \mathcal{H}(D^n)$
$--$	$--$	$\pi_0 \mathscr{C}(D^n)$	$\pi_1 \mathscr{C}(D^n)$	$\pi_2 \mathscr{C}(D^n)$	$\pi_3 \mathscr{C}(D^n)$	$\pi_4 \mathscr{C}(D^n)$
$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0

Table 2: Comparison of Homotopy Groups ($i = 7$ to 13)

$\pi_7 \mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_8 \mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_9 \mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_{10} \mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_{11} \mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_{12} \mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_{13} \mathrm{Wh}^{\mathrm{Diff}}(*)$
$\pi_6 \mathcal{H}(D^n)$	$\pi_7 \mathcal{H}(D^n)$	$\pi_8 \mathcal{H}(D^n)$	$\pi_9 \mathcal{H}(D^n)$	$\pi_{10} \mathcal{H}(D^n)$	$\pi_{11} \mathcal{H}(D^n)$	$\pi_{12} \mathcal{H}(D^n)$
$\pi_5 \mathscr{C}(D^n)$	$\pi_6 \mathscr{C}(D^n)$	$\pi_7 \mathscr{C}(D^n)$	$\pi_8 \mathscr{C}(D^n)$	$\pi_9 \mathscr{C}(D^n)$	$\pi_{10} \mathscr{C}(D^n)$	$\pi_{11} \mathscr{C}(D^n)$
$\mathbb{Z}_2$	(0?)	$\mathbb{Z}_2 \oplus \mathbb{Z}$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_8$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	$\mathbb{Z}_4 \oplus (0?)$	$\mathbb{Z} \oplus \mathbb{Z}_7$

Table 3: Comparison of Homotopy Groups ($i = 14$ to 19)

$\pi_{14}\mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_{15}\mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_{16}\mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_{17}\mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_{18}\mathrm{Wh}^{\mathrm{Diff}}(*)$	$\pi_{19}\mathrm{Wh}^{\mathrm{Diff}}(*)$
$\pi_{13}\mathcal{H}(D^n)$	$\pi_{14}\mathcal{H}(D^n)$	$\pi_{15}\mathcal{H}(D^n)$	$\pi_{16}\mathcal{H}(D^n)$	$\pi_{17}\mathcal{H}(D^n)$	$\pi_{18}\mathcal{H}(D^n)$
$\pi_{12}\mathscr{C}(D^n)$	$\pi_{13}\mathscr{C}(D^n)$	$\pi_{14}\mathscr{C}(D^n)$	$\pi_{15}\mathscr{C}(D^n)$	$\pi_{16}\mathscr{C}(D^n)$	$\pi_{17}\mathscr{C}(D^n)$
$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus (0?)$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_{32} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$	$\mathbb{Z}_2 \rtimes \mathbb{Z}_2 \rtimes \mathbb{Z}_8 \rtimes \mathbb{Z}_2 \oplus \mathbb{Z}_7$

References

[Rog02] John Rognes. Two-primary algebraic K-theory of pointed spaces. *Topology*, 41(5):873–926, September 2002.

[Rog03] John Rognes. The smooth Whitehead spectrum of a point at odd regular primes. *Geometry & Topology*, 7(1):155–184, March 2003. arXiv:math/0304384.

[Wal82] Waldhausen. Algebraic K-theory of spaces, a manifold approach. *Canadian Mathematical Society Conference Proceedings*, 2, 1982.

[Wei] Charles Weibel. Algebraic K-Theory I.5 of Rings of Integers in Local and Global Fields.